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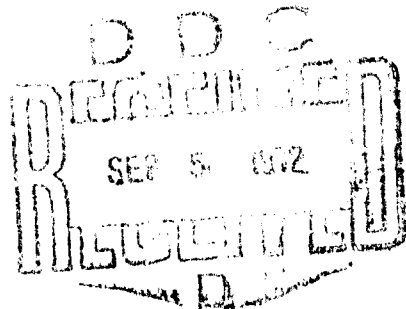
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HDL-TR-1584

PULSE-POSITION MODULATION  
FOR SIGNAL IDENTIFICATION

by  
Philip B. Kaplan

March 1972



U.S. ARMY MATERIEL COMMAND  
**HARRY DIAMOND LABORATORIES**  
WASHINGTON, D.C. 20438

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<p>The use of pulse position modulation (PPM) to code N signals has been investigated. The following three coding schemes were considered: (1) identical pulse periods with synchronization, (2) distinct pulse periods without synchronization, and (3) position modulation of alternate pulses. The theoretical performance of these methods was considered with respect to the probability of overlapping signals, the average frequency of such overlaps, and the generation of false signals.</p> <p>This investigation was concerned primarily with performance limitations due to signal design rather than receiver characteristics; thus, only the properties of the signal in an infinite signal-to-noise ratio environment were considered. The false signal was found to be non-existent in coding method (1); and for each of the other two schemes, false signals were found to occur less frequently than pulse overlap. For any given set of parameters, the performance of each coding scheme as measured by the frequency of overlaps was found to be the same.</p>			

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## 1. INTRODUCTION

The use of pulse-position modulation (PEM), including variation of the pulse period, has been investigated by the Harry Diamond Laboratories as a method of obtaining signal identification. This study was conducted for Research and Engineering Directorate of the U.S. Army Missile Command.

Basically, this study considers a set of  $N$  periodic pulse trains, each emanating from a distinct transmitter or source as shown in figure 1. The objective of this study is to determine a means of coding each pulse train in a way to make it possible for a receiver to recognize a particular pulse train with a low probability of error. Modulation and receiver costs, as well as the time to decode, are not considered in this study; ultimately, however, these factors must be considered.

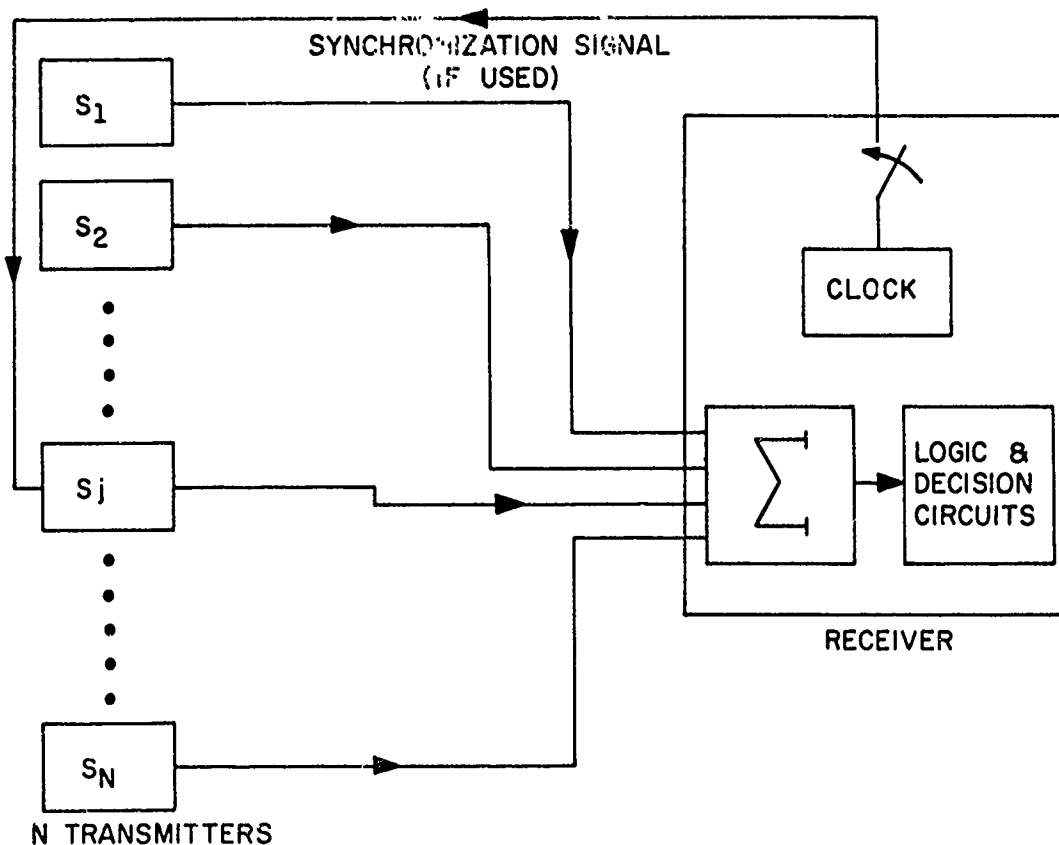


Figure 1. Model of coding problem.

Regardless of the receiver that is actually used, the performance of the system is limited by the properties of the signals  $S_1, \dots, S_N$  that are transmitted. If there are overlapping signals or if the sum of two signals appears to be a third signal, any receiver will be restricted in its ability to decode or separate the signals. Accordingly, performance limitation due to signal design is the primary concern in this report rather than the characteristics of the receiver.

Each transmitter puts out a pulse train of approximately the same frequency, with the only allowable modulation being a small displacement in pulse position. Each pulse in all pulse trains is the same shape.

Treating time as discrete greatly facilitates an analysis of this type of problem. Such an approach is possible if the maximum effect of the timing uncertainties--which arise from such parameters as finite pulse width, error in synchronization signal, and receiver resolution ability--is small compared with the period of each pulse train. This study assumes that such an approach is valid, and time will be taken as discrete. The time grain  $t_0$  seconds will account for all the timing uncertainties;  $t_0$  will be called the resolution cell, and the location of a particular pulse in any pulse train will always be given as the number of the resolution cell in which it resides. In this way, each pulse train can be thought of as a sequence of zero's and one's.

## 2. MODULATION METHODS

### 2.1 Identical Period Pulse Trains with Synchronization

Considered here are  $N$  pulse trains,  $S_i$ ,  $i=1, 2, \dots, N$ , each with the period  $\tau$  and one pulse per period. The only distinguishing feature of any pulse train is the location of the pulse within each interval of length  $\tau$  resolution cells. Hence, the only method of coding or identifying the signals to make them recognizable to the receiver is synchronization. It is assumed that synchronization has been established only between the receiver and transmitter of the desired signal. The location of each pulse in the  $i$ th pulse train is given by  $n\tau + t_i$  for some  $n$ , where  $t_i$  is a fixed integer in the interval  $[1, \tau]$ . The receiver is looking for the  $j$ th pulse train and, hence, knows  $t_j$ . It is the task of the receiver not only to receive the  $j$ th pulse train, but to lock onto it. Precisely, this means the receiver must be able to identify  $S_j$ . Knowing  $t_j$ , the receiver will always be able to identify  $S_j$ , the only occasion for error being when the receiver identifies  $S_k$  as  $S_j$  for  $k \neq j$ . This condition occurs only if  $t_j = t_k$  for some  $k \neq j$ .

The period of each pulse train is the same; therefore, if an overlap occurs, it will occur every period. Thus, it is necessary to examine the pulse trains for only  $\tau$  resolution cells to perform signal identification. Note that it is also possible for more than one signal to overlap the desired pulse train.

Let  $A_N$  be the event that none of the  $N$  pulse trains overlap. If  $A_N$  occurs, the receiver will be able to function properly--that

is, receive and lock onto the specified pulse train. The probability of this event,  $P(A_N)$ , is now calculated.

Since numbering of the pulse trains is arbitrary, it may be assumed without loss of generality that the receiver wishes to lock onto the first pulse train. It is assumed that the starting times of the  $N$  pulse trains are independent, uniformly distributed random variables. The location of the pulse that lies in the interval  $[1, \tau]$  of pulse train number one is some integer  $k$ . The probability that the pulse of  $S_2$  in the interval  $[1, \tau]$  does not occur in resolution cell  $k$  is  $(\tau-1)/\tau$ . Similarly, given that  $S_1$  and  $S_2$  do not overlap, the probability that  $S_3$  does not overlap  $S_1$  or  $S_2$  is  $(\tau-2)/\tau$ . By induction, for  $N$  signals,

$$P(A_N) = \prod_{i=1}^{N-1} \frac{\tau - i}{\tau} \quad (1)$$

The expression  $1 - P(A_N)$  is plotted in figure 2 for  $2 \leq N \leq 10$ . It can be seen that for large values of  $\tau$ --that is, many resolution cells per period--the probability of overlap between any of the signals is small, even for relatively large numbers of signals. One way to increase  $\tau$  for fixed time grain  $t_0$  is to decrease the pulse repetition frequency. This would increase  $P(A_N)$ . This would, however, affect the time to decode. Hence, there may be a lower limit on the pulse repetition frequency. Furthermore,  $t_0$  will probably be determined by the operating environment.

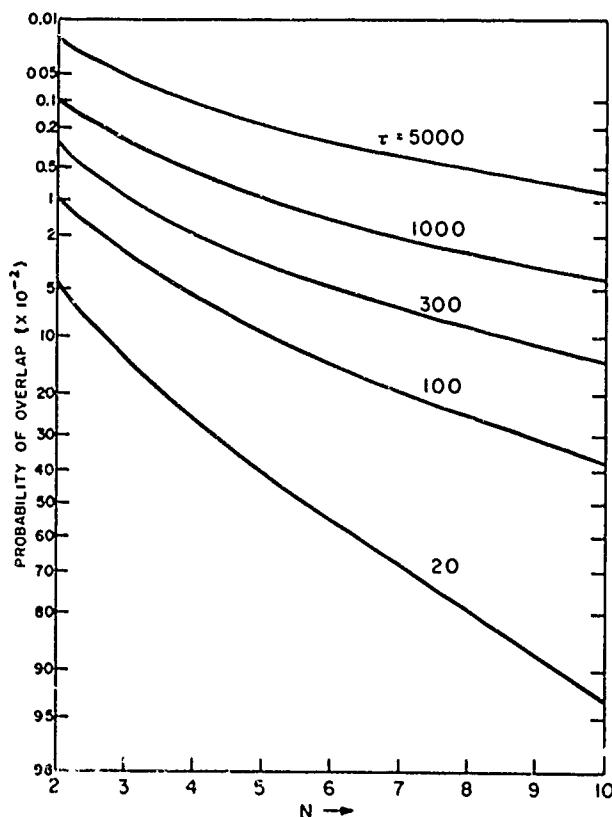


Figure 2. Probability of overlap versus  $N$  for different values of  $\tau$ .

By specifying an acceptable value of  $P(A_N)$ , one will then be able to determine  $N$  for any given value of  $\tau$ .

The above analysis assumes that the receiver is able to communicate with the transmitter of the desired signal to the extent necessary to achieve synchronization. If it is further assumed that the receiver can establish synchronization with the other  $N-1$  signals, each pulse train can be set so that no overlapping occurs. Then for  $N \leq \tau$ ,  $P(A_N) = 1$  and the system will perform perfectly.

## 2.2 Periodic Pulse Trains with Distinct Periods

In this problem, there are  $N$  pulse trains or signals  $\{S_i\}_{i=1}^N$ , each with period  $\tau_i$  resolution cells, and one pulse per period. The numbers  $\{\tau_i\}$  will be taken to be close to one another, but distinct unless stated otherwise. Again, the task of the receiver is to identify one pulse train and lock onto it. The situation here is more difficult to analyze than that of the first case (sec. 2.1) since each pulse train now has a different period.

Consider two signals  $S_i$  and  $S_j$ ,  $i \neq j$ , with periods  $\tau_i$  and  $\tau_j$ , respectively, and let  $S_0$  be the logical AND of  $S_i$  and  $S_j$ . The period  $\tau_0$  of  $S_0$  is the least common multiple<sup>1</sup> (LCM) of  $\tau_i$  and  $\tau_j$ , that is,

$$\tau_0 = \text{LCM}(\tau_i, \tau_j) \quad (2)$$

By the properties of the LCM there exists integers  $n_i$  and  $n_j$  such that

$$\tau_0 = n_i \tau_i = n_j \tau_j \quad (3)$$

Let  $B$  be the event that an overlap occurs, in which case  $S_0$  contains other than all zero's. In time  $\tau_0$ , there are  $n_i$  pulses of  $S_i$  and  $n_j$  pulses of  $S_j$ . Since  $\tau_0$  is the LCM of  $\tau_i$  and  $\tau_j$ , every possible overlap condition involves exactly one overlap. Hence, there are  $n_i n_j$  distinct ways that overlap can occur. Again, assuming uniformly distributed independent starting positions for each pulse train, it is seen that

$$P(B) = \frac{n_i n_j}{\tau_0} \quad (4)$$

From (3),

$$P(B) = \frac{n_i}{\tau_j} = \frac{n_j}{\tau_i} \quad (5)$$

It is desirable to compare case 2 (distinct pulse periods) with case 1 (sec. 2.1). When  $P(B)$  is calculated, an interesting phenomenon is observed. (This was pointed out by V.J. Graham, formerly of HDL.)

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<sup>1</sup> I. Herstein, *Topics in Algebra*, Blaisdell Publishing Co., New York, 1964, p. 22.

Suppose  $\tau_i$  and  $\tau_j$  are relatively prime, that is, contain no common factors. Then

$$\tau_0 = \tau_i \tau_j \quad (6)$$

$$n_i = \tau_j \quad (7a)$$

and

$$n_j = \tau_i \quad (7b)$$

Hence, (4) implies that  $P(B) = 1$ . There will always be an overlap somewhere regardless of the relative starting positions of the two signals. If two out of the  $N$  signals are relatively prime, then the probability of no overlap between any of the  $N$  signals is zero. From equation (4) if  $\tau_i$  and  $\tau_j$  are not relatively prime, then  $P(B) < 1$ .

Several anomalies are apparent. First is the fact that  $P(B)$  is highly dependent upon whether or not  $\tau_i$  and  $\tau_j$  are relatively prime. Thus, the performance of the system as measured by  $P(B)$  is sensitive to a quantity that does not seem to be physically significant. Since the  $N$  signals have approximately the same period, it may reasonably be assumed that there are two signals whose periods are consecutive integral multiples of the resolution cell time.

proposition: Let  $m$  and  $n$  be two positive integers such that  $m = n + 1$ . Then  $m$  and  $n$  are relatively prime.

proof: Suppose the proposition is false. Then there exist integers  $k, p, q$ , with  $k \geq 2$ , such that  $kp = n$  and  $kq = m$ . Therefore, both  $n/k$  and  $n/k + 1/k$  are integers, implying that  $1/k$  is an integer. Since  $k \geq 2$ , this is a contradiction and the proposition is proved. end.

From the proposition and the above discussion, it follows that for  $N$  signals of nearly the same period there will always be an overlap. Thus, the probability of overlap does not seem to be a satisfactory measure of system performance.

In the case of  $N$  signals with identical periods, whatever happens in one period happens in every period. In the present situation of distinct pulse periods, the interaction between pulse trains is not the same from period to period. Hence, rather than considering the probability of a given event ever occurring, it is more significant to consider the frequency with which it happens.

For two signals of a given initial relative starting position, the signals will either not overlap, or overlap periodically. For two signals  $S_1$  and  $S_2$ , let  $f_{av}$  be the average frequency of overlap where the expectation is taken over the possible relative starting positions. That is, for each of the  $J$  possible relative starting positions, let  $f_i$  be frequency of overlap, where  $f_i = 0$  if no overlap occurs. Let  $C_i$  be the event that the  $i^{th}$  relative starting position occurs. Then

$$f_{av} = \sum_{i=1}^J f_i P[C_i] \quad (8)$$

Now,  $f_{av}$  will be calculated in terms of the pulse periods. There are two cases to consider. First, suppose  $\tau_1$  and  $\tau_2$  are relatively prime. Then the logical AND of  $S_1$  and  $S_2$  has a period of  $T = \tau_1 \tau_2$ . Hence, a length  $T$  of this joint pulse train is considered. Let  $t_1 \in [1, \tau_1]$  and  $t_2 \in [1, \tau_2]$  be the location of the "first" pulses of  $S_1$  and  $S_2$ , respectively.

The pulses of  $S_i$ ,  $i = 1, 2$ , in  $[1, t]$  occur at  $n_i \tau_i + t_i$ ,  $n_j = 0, 1, \dots, T/\tau_i - 1$ , where  $n_i$  is any integer. It is desired to find the positions where  $S_1$  and  $S_2$  overlap, that is,  $n_1$  and  $n_2$ , so that

$$n_1 \tau_1 + t_1 = n_2 \tau_2 + t_2 \quad (9)$$

lemma: Let  $a$  and  $b$  be two integers whose greatest common divisor is  $g$ . Then there exists two integers  $p$  and  $q$  such that

$$pa + qb = g \quad (10)$$

proof: Herstein,<sup>1</sup> p. 17. end

theorem: Let  $\tau_1$  and  $\tau_2$  be relatively prime integers,  $1 \leq t_1 \leq \tau_1$  and  $1 \leq t_2 \leq \tau_2$ . Then equation (9) possesses a solution  $(n_1, n_2)$  and each solution of equation (9) is of the form  $(k\tau_2 + n_1, k\tau_1 + n_2)$ .

proof: Rewriting equation (9) gives

$$n_1 \tau_1 - n_2 \tau_2 = t_2 - t_1 \quad (11)$$

Let  $a = \tau_1$  and  $b = \tau_2$ . Since  $\tau_1$  and  $\tau_2$  are relatively prime, the lemma implies that integers  $p$  and  $q$  exist such that

$$p\tau_1 + q\tau_2 = 1 \quad (12)$$

Assuming  $t_1 \neq t_2$  and multiplying equation (12) by  $t_2 - t_1$  gives equation (11), with  $n_1 = p(t_2 - t_1)$  and  $n_2 = q(t_1 - t_2)$ . If  $t_1 = t_2$ , then equation (11) is satisfied with  $n_1 = 0 = n_2$ .

It has been shown that a solution  $(n_1, n_2)$  for equation (11) exists. The general form of the solution is now determined.

Suppose that equation (11) is satisfied by  $(n_1, n_2)$  and  $(n_3, n_4)$ . Then

$$n_1 \tau_1 - n_2 \tau_2 = n_3 \tau_1 - n_4 \tau_2 \quad (12)$$

or

$$(n_1 - n_3) \tau_1 = (n_2 - n_4) \tau_2 \quad (13)$$

<sup>1</sup> I. Herstein, *Topics in Algebra*, Blaisdell Publishing Co., New York, 1964, p. 17.

Therefore, dividing equation (13) by  $\tau_1$  and observing that  $\tau_1$  and  $\tau_2$  are relatively prime implies that  $\tau_1$  divides  $n_2 - n_4$ , or

$$n_4 = k_1 \tau_1 + n_2 \quad (14a)$$

Similarly,

$$n_3 = k_2 \tau_2 + n_1 \quad (14b)$$

From equation (12), it is seen that  $k_1 = k_2$ .

Therefore, the general form of the solution to equation (11) is  $(k\tau_2 + n_1, k\tau_1 + n_2)$ , for all  $k$  integer. end

The above theorem shows that, for every relative starting position  $C_i$  of the two signals  $S_1$  and  $S_2$  whose periods  $\tau_1$  and  $\tau_2$  are relatively prime, there is precisely one overlap per time  $\tau_1 \tau_2$ . Hence, for each  $i$ ,  $f_i = 1/(\tau_1 \tau_2)$ ,  $P[C_i] = 1/J$ , and equation (8) becomes

$$f_{av} = \frac{1}{\tau_1 \tau_2} \quad (15)$$

It is now desired to calculate  $f_{av}$  for the case where  $\tau_1$  and  $\tau_2$  are not relatively prime. Let  $c > 1$  be the largest common factor of  $\tau_1$  and  $\tau_2$ . Let  $r_1$  and  $r_2$  be such that

$$\tau_1 = cr_1 \quad (16a)$$

$$\tau_2 = cr_2 \quad (16b)$$

If the proof of the theorem is now repeated, equation (11) becomes

$$n_1 cr_1 - n_2 cr_2 = c(n_1 r_1 - n_2 r_2) = t_2 - t_1$$

However,  $t_2 - t_1$  may or may not be divisible by  $c$ . Suppose that  $t_2 - t_1$  contains  $c$  as a factor. Then equation (17) can be divided by  $c$ , and since  $r_1$  and  $r_2$  are relatively prime, the theorem implies that there precisely one overlap every  $cr_1 r_2$  time unit. Since the left-hand side of equation (17) is divisible by  $c$ , equation (17) will possess a solution only if  $t_2 - t_1$  is divisible by  $c$ . Hence, for the relative starting positions of  $S_1$  and  $S_2$  so that  $t_2 - t_1$  is not divisible by  $c$ ,  $S_1$  and  $S_2$  will never overlap, and  $f_i = 0$  for these configurations. Without loss of generality,  $t_1$  may be taken to zero;  $t_2$  is a uniformly distributed, integer-valued, random variable. Since one of every  $c$  integers is divisible by  $c$ , the probability of  $t_2 - t_1$  being divisible by  $c$  is  $1/c$ . Therefore,  $f_{av}$  is given by (8) as

$$\begin{aligned} f_{av} &= \frac{1}{cr_1 r_2} \frac{1}{c} \\ &= \frac{1}{\tau_1 \tau_2} \end{aligned} \quad (18)$$

Thus, it is seen from equations (15) and (18) that  $f_{av}$  is given by the same expression, whether or not  $\tau_1$  and  $\tau_2$  are relatively prime. This is intuitively appealing, since one would not expect the performance of a physical system to depend strongly on the property of relative primeness of two quantities.

If the receiver must be able to identify signals rapidly,  $f_{av}$  will be required to be a small number. If more than two signals are involved, the relative frequency of three or more signals overlapping at a point in time will be much smaller than the frequency of overlap for any two signals and is neglected here. Therefore, as a measure of system performance for  $N$  signals, the quantity  $F$  is defined to be the sum of the average frequency of overlap for every pair of signals  $S_i$  and  $S_j$ ,  $i \neq j$ , where  $1 \leq i \leq N$  and  $1 \leq j \leq N$ . If  $F$  is small, overlapping of signals is infrequent. If  $F$  is large, the overlapping of signals will be a serious problem and  $F$  fails to be a meaningful measure of overlap.

It was stated earlier that the periods of the  $N$  signals are approximately equal. Therefore, suppose that for  $N$  signals  $\{S_i\}$ , the period  $\tau_i$  of  $S_i$  can be written as

$$\tau_i = \bar{\tau} + \epsilon_i \quad i = 1, 2, \dots, N \quad (19)$$

where  $\bar{\tau}$  and  $\epsilon_i$  are both integers, and

$$\epsilon_i \ll \bar{\tau} \quad i = 1, 2, \dots, N \quad (20)$$

Then, for signal  $S_i$  and  $S_j$ ,

$$\begin{aligned} f_{av} &= \frac{1}{(\bar{\tau} + \epsilon_i)(\bar{\tau} + \epsilon_j)} \\ &\approx \frac{1}{\bar{\tau}^2} \end{aligned} \quad (21)$$

Thus,  $f_{av}$  for two signals of this type is essentially independent of the particular pair of signals. Since the number of signal pairs that can be formed from  $N$  signals is given by

$$C_2^N = \frac{N(N-1)}{2} \quad (22)$$

it follows that  $F$  is given by

$$F = \frac{N(N-1)}{2\bar{\tau}^2} \quad (23)$$

Suppose there are  $N$  signals with a nominal period of  $\bar{\tau} = 100$  resolution cells. Then,

$$F = 5(10^{-5}) N(N-1) \quad (24)$$

For  $N = 2$ ,  $F = 10^{-4}$ , which means there is an average of one overlap every  $10^4$  resolution cells. In this case, that means one overlap every 100 periods. Suppose now that  $N = 5$ . Then equation (24) implies  $F = 10^{-3}$ , or that on the average there is one overlap somewhere every 10 periods.

These results merit some discussion. It is beyond the scope of this report to say whether any particular value of  $F$  renders a system either practical or infeasible. Answers to such questions depend upon both the application of the system and the reliability with which the task is to be performed and the "acceptable" cost. In any case, the value of  $F$  must be interpreted correctly. In the second numerical example, it was found that among the five signals, there is on the average one overlap per 10 periods. Now, for each pair of the five signals, there may be either no overlap or one overlap every  $\bar{T}$  resolution cells for some  $\bar{T}$ . As  $\bar{T}$  decreases, the probability of an overlap becomes smaller. It is possible that two signals can overlap every pulse period or every other pulse period. Although this happens relatively infrequently, if it is occurring with the signal that the receiver is attempting to lock onto, it is an intolerable situation. For any particular value of  $f_{av}$ , it is not possible to say whether this value results from the rare occurrence of frequent overlaps or the highly likely case of infrequent overlaps. The point is that, using random process terminology,  $F$  represents an ensemble average, the system performance is dependent on a time average, and the process is not ergodic. The usefulness of  $F$  as a performance measure lies in the fact that for small values of  $F$ , the probability of a high overlap rate for any pair of signals is small. Thus, it is unlikely that the system will fail because of overlaps if  $F$  has a small value.

The performance of identical pulse periods and possibly distinct pulse periods is now compared. If there is ever an overlap with  $N$  identical pulse periods, there will be an overlap every period. Therefore, the probability of no overlap is a useful performance measure for this system. For comparison purposes, the probability of overlap is now related to the total average frequency of overlap for  $N$  identical pulse periods.

Let the period of  $S_i$ ,  $i = 1, 2, \dots, N$  be  $\bar{\tau}$  resolution cells. Then, using the same argument as above,

$$f_{av} = \frac{1}{\bar{\tau}^2} \quad (25)$$

and

$$F = \frac{N(N-1)}{2} \frac{1}{\bar{\tau}^2} \quad (26)$$

It is seen that both systems--that is, identical pulse periods and distinct pulse periods--have the same performance as measured by  $F$ . However, as stated above,  $F$  does not completely specify system performance. As a simple example, consider the following two possible configurations:

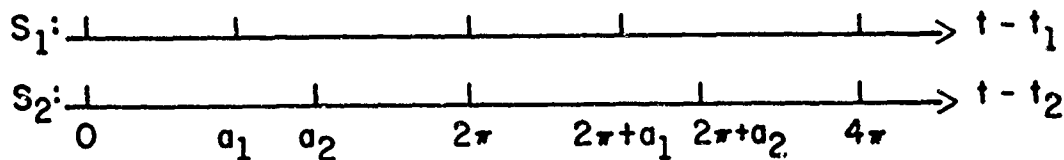


Figure 3. Two pulse trains.

$$(a) N = 2, \tau_1 = \tau_2 = 100,$$

$$(b) N = 2, \tau_1 = 99, \tau_2 = 100.$$

$F = (10^{-4})$  for both cases. In case (a), the probability that the signals overlap is 0.01, but if overlaps do occur, they do so every pulse period--that is, every 100 resolution cells. Thus, the signals will overlap completely and the system will not function one percent of the time. On the other hand, in case (b) the signals will always overlap, but there will be approximately one overlap every 100 pulse periods or  $10^4$  resolution cells. It is seen that although  $F$  is about the same for both systems, the performance characteristics are quite distinct.

Suppose the probability of the receiver successfully finding the desired pulse train exceeds 99 percent when it loses one pulse from each 100, then B is the desired operating configuration. Otherwise, configuration A is preferred since in this case the system works 99 percent of the time (assuming, of course, that the system always works if it receives 100 percent of the pulses). Hence, either system may be preferable, depending upon the situation.

### 2.3 Alternate Pulse Modulation

In this coding scheme, every other pulse is modulated. More precisely, there are  $N$  signals  $S_i$ ,  $i = 1, 2, \dots, N$ , each with the same period  $2\tau$ . In time, the locations of the pulses in  $S_i$  are of the form

$$2n\tau + t_i ,$$

$$2n\tau + t_i + a_i , \quad (27)$$

where  $t_i$  is the initial phase and  $a_i \neq a_j$  for  $i \neq j$ . Two pulse trains are illustrated in figure 3.

Since  $a_i \neq a_j$  for  $i \neq j$ ,  $S_i$  and  $S_j$  either never overlap or overlap every other pulse which is once a period. Thus, alternate pulse modulation in some sense can be considered to lie between identical period pulse trains and periodic pulse trains with distinct periods. It is similar to the identical period case, in that it is periodic and thus a meaningful probability of overlap may be calculated. It is like distinct pulse periods in that each pulse train is distinguishable without synchronization and it is not possible for two pulse trains to overlap one another for all pulses.

The probability of no overlap  $C_N$  is now calculated. Consider  $S_i$  with pulses at  $t_i$  and  $t_i + a_i$  for  $i = 1, 2$ .

Since  $S_1$  and  $S_2$  can overlap only once per period, there are four values for  $t_2$  that will produce an overlap. Hence,

$$\begin{aligned} P[C_2] &= \frac{2\tau - 4}{2} \\ &= \frac{\tau - 2}{\tau} \end{aligned} \quad (28)$$

Suppose now that  $S_1$  and  $S_2$  do not overlap and that the probability of  $S_3$  overlapping  $S_1$  and  $S_2$  is negligibly small. Then there are eight values of  $t$  that will produce an overlap and, hence,

$$P[C_N] = \frac{\tau - 2}{\tau} \frac{\tau - 4}{\tau} \quad (29)$$

By induction,

$$P[C_N] = \prod_{j=1}^{N-1} \frac{\tau - 2^j}{\tau} \quad (30)$$

If  $2^{N-1} > \tau$ , then  $P[C_N]$  as given by equation (30) is clearly not valid. This is because of the assumption regarding multiple overlap. However, for the system to be usable,  $P[C_N]$  will be near unity and therefore equation (30) is an adequate approximation. The expression  $1 - P[C_N]$  is plotted in figure 4. In comparing figures 4 and 2, it should be noted that  $\tau$  represents the average number of resolution cells between pulses for both systems. Regarding overlap, it can be seen that the system employing  $N$  identical pulse periods is superior to that of alternate pulse modulation.

The total average frequency of overlap is now calculated. The probability of overlap for two signals is  $2/\tau$  and frequency of such an overlap is  $1/(2\tau)$ . Therefore,

$$f_{av} = \frac{1}{\tau^2} \quad (31)$$

and

$$F = \frac{N(N-1)}{2\tau^2} \quad (32)$$

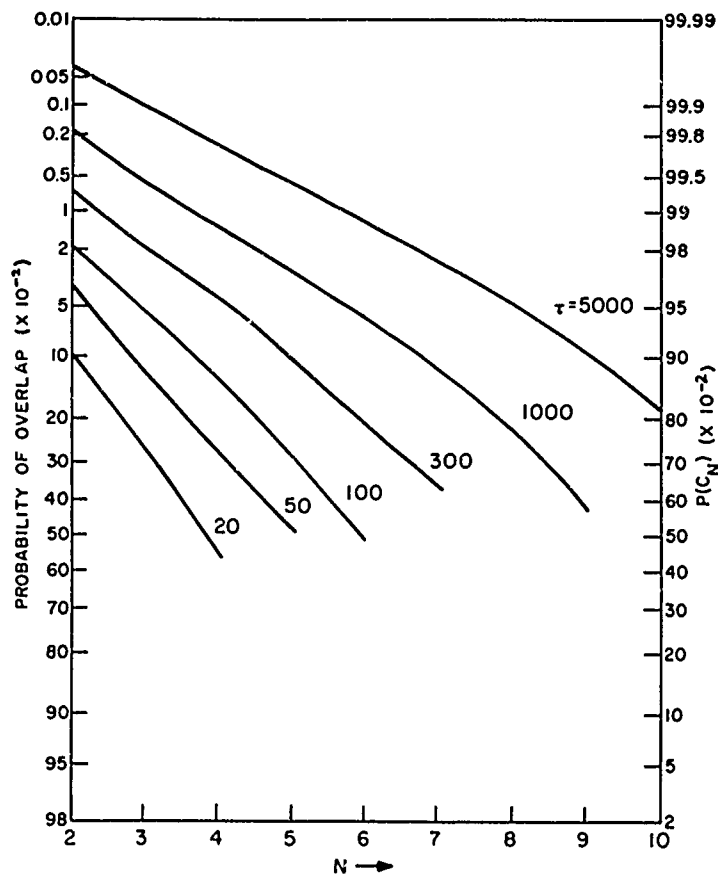


Figure 4. Probability of  $P(C_N)$  versus  $N$  for different values of  $\tau$ .

It is seen from equations (23), (26), and (32) that all three systems have the same value of  $F$ . Since equations (23) and (26) are the same and alternate pulse modulation lies "between" identical pulse period and distinct pulse periods, equation (32) is not unexpected.

#### 2.4 False Signals

The above sections have been concerned with the problem of overlapping signals in which one signal may hide the existence of another signal. Another possible type of "error" is that of two or more undesired signals arriving at the receiver at such times so as to appear to be the desired signal. This phenomenon will be referred to as the generation of a false signal.

In the case of  $N$  signals with identical pulse periods, the receiver is synchronized with the desired signal. The only type of error is that of overlapping signals. It is not possible to generate a false signal unless the receiver is synchronized to the wrong signal. But this possibility has been ruled out by hypothesis.

For  $N$  signals with distinct pulse periods, it is possible to generate false signals. The mechanism by which this happens and the effect on system performance depend heavily on the logic of the receiver. Suppose that the receiver is seeking signal  $S_i$ . It examines all of the incoming pulses in an attempt to find a pulse train with period  $\tau_i$ . The question then arises as to how long the receiver should look. It will be assumed that there is a cost associated with this decision time. In fact, the cost may rise in a highly nonlinear manner with increasing decision time. Clearly, the longer the receiver looks, the less likely it is that it will lock onto the wrong signal. For example, suppose the receiver requires two pulses located  $\tau_i$  time units apart to say it has found  $S_i$ . For a false detection to occur, two false pulses--each from a distinct pulse train--are required to be in a certain relationship to each other. This is roughly equivalent to the requirement for two signals to overlap. Thus, both the probability and relative frequency of these two events (overlap and false signals) are about the same. It is much more improbable that three incorrect pulses separated by  $\tau_i$  will occur. Thus, if the decision time is  $3\tau_i$  and the frequency of overlap is small, then the possibility of false codes is negligibly small. The choice of a decision time depends on the cost of decision time and the complexity of the decision scheme that one is willing to use; in turn, these factors depend upon the particular application.

Suppose now that alternate pulse modulation is being employed, and the receiver is seeking signal  $S_j$ . Then it is possible for two undesired signals to be arranged to appear as  $S_j$  to the receiver. The probability of this occurrence is the same as the probability of overlap. Hence, alternate pulse modulation is at least as likely to suffer from false signal interference as for distinct pulse period modulation.

### 3. SUMMARY AND CONCLUSIONS

This report discusses the problem of coding a finite collection of pulse trains in a manner that permits the identification of any single pulse train specified. Three particular schemes were considered--namely: identical pulse periods with synchronization, distinct pulse periods without synchronization, and modulation of the position of every other pulse. Factors such as cost of decoding and complexity of implementation were not considered, although these parameters would ultimately affect the choice of a particular scheme for a given application.

Two phenomena have been examined--pulses from two or more signals overlapping in time so as to be indistinguishable and two erroneous signals combining in a way to produce a false signal that appears to be the desired signal. The false signal was found to be non-existent in the case of identical pulse periods; and for each of the other two schemes, false signals were found to occur less frequently than pulse overlap.

The average frequency of overlap,  $f$ , where the average is taken over all the possible relative starting positions, was introduced. It was shown that for each of the three coding schemes,  $f$  is given by approximately the same expression. Thus, for any given set of parameters, the performance of each coding scheme as measured by  $f$  is the same. However, since  $f$  is an average quantity, it is possible for different sets of signals having different overlap characteristics to give the same value of  $f$ . Even though  $f$  is only an average, and does not completely describe system performance, it can be stated that for sufficiently small  $f$ , the likelihood of difficulties arising from overlaps is nearly eliminated.